Strongly Monolithic Characters of Finite Groups

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Group Representations and Characters

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Let ϑ be a representation of G afforded by $\mathbb{C}[G]$-modul V. We say that ϑ is an irreducible representation of G if V is an irreducible $\mathbb{C}[G]$-modul.

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Definition

Let χ be a character afforded by the representation ϑ of G. χ is called an irreducible character if ϑ is an irreducible representation of G.

Group Representations and Characters

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- Let χ be a character of G afforded by the representation ϑ. Then the kernel of χ, write $\ker\chi$, is the kernel of ϑ. $\ker\chi = \{g \in G \mid \chi(g) = \chi(1)\}$.

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- Let χ be a character of G afforded by the representation ϑ. Then the kernel of χ, write $\ker\chi$, is the kernel of ϑ. $\ker\chi = \{g \in G \mid \chi(g) = \chi(1)\}$.
- We say that χ is a faithful character of G if $\ker\chi = 1$.

Strongly Monolithic Characters

Definition

Let G be a group and let χ be an irreducible character of G. If $G/\ker\chi$ has only one minimal normal subgroup then the irreducible character χ is called a monolithic character.

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- $Z(\chi) = \ker\chi$ where $Z(\chi) = \{g \in G \mid |\chi(g)| = \chi(1)\}$,
- $G/\ker\chi$ is a p-group whose commutator subgroup is its unique minimal normal subgroup.

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Theorem (I.M. Isaacs, Character Theory of Finite Groups)

Let G be solvable and assume that G' is the unique minimal normal subgroup of G. Then all nonlinear irreducible characters of G have equal degree f and one of the following situations obtains:

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- *G is p-group, $Z(G)$ is cyclic and $G/Z(G)$ is elementary abelian of order f^2.*
- *G is a Frobenius group with an abelian Frobenius complement of order f. Also, G' is the Frobenius kernel and is an elementary abelian p-group.*

Strongly Monolithic Characters

Character table of $SL_2(\mathbb{F}_3)$

class	1	2	3A	3B	4	6A	6B	
size	1	1	4	4	6	4	4	
ρ_1	1	1	1	1	1	1	1	trivial
ρ_2	1	1	ζ_3^2	ζ_3	1	ζ_3^2	ζ_3	linear of order 3
ρ_3	1	1	ζ_3	ζ_3^2	1	ζ_3	ζ_3^2	linear of order 3
ρ_4	2	-2	-1	-1	0	1	1	symplectic faithful, Schur in
ρ_5	2	-2	ζ_6^5	ζ_6	0	ζ_3	ζ_3^2	complex faithful
ρ_6	2	-2	ζ_6	ζ_6^5	0	ζ_3^2	ζ_3	complex faithful
ρ_7	3	3	0	0	-1	0	0	orthogonal lifted from A_4

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Theorem (Erkoç, Güngör, Özkan, 2022)

Let G be a solvable group and let χ be a primitive monolithic character of G. Then χ cannot be a strongly monolithic character of G.

Strongly Monolithic Characters

Character table of A_5

class	1	2	3	5A	5B	
size	1	15	20	12	12	
ρ_1	1	1	1	1	1	trivial
ρ_2	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	orthogonal faithful
ρ_3	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	orthogonal faithful
ρ_4	4	0	1	-1	-1	orthogonal faithful
ρ_5	5	1	-1	0	0	orthogonal faithful

Literature

Theorem (Thompson, 1970)

Let G be a group. Suppose p $\mid \chi(1)$ *for every nonlinear irreducible character* χ *of G, where p is prime. Then G has a normal p-complement.*

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Theorem (Ito-Michler)

Let G be a finite group and let p be a prime divisor of $|G|$. Then p does not divide $\chi(1)$ for every nonlinear irreducible character χ of G if and only if G has a normal abelian Sylow p-subgroup.

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Theorem (Isaacs ve Passman, 1968)

Let G be a finite group. If $cd(G) - \{1\}$ is subset of the set of prime numbers, then G is solvable.

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Theorem (Taketa, 1930)

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Monomial groups are solvable.

Theorem (Isaacs, 1969)

Let G be a finite group. If $|cd(G)| \leq 3$*, then G is solvable.*

Literature

Theorem (Berkovich, Zhmud, Character Theory of Finite Groups (Part 2), 1997)

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- *If $p \mid \chi(1)$ for every nonlinear monolithic character χ of G, then G has a normal p-complement.*

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Let G be a finite group and let p be a prime number.

- *If* $p \mid \chi(1)$ *for every nonlinear monolithic character* χ *of G, then G has a normal p-complement.*
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- *$p \nmid \chi(1)$ for every nonlinear monolithic character χ of G if and only if G has a normal abelian Sylow p-subgroup.*
- *If $\chi(1)$ is a prime for every nonlinear monolithic character χ of G, then G is solvable.*

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- *If $\chi(1)$ is a prime for every nonlinear monolithic character χ of G, then G is solvable.*
- *If every monolithic character of G is monomial, then G is solvable.*

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- *If $\chi(1)$ is a prime for every nonlinear monolithic character χ of G, then G is solvable.*
- *If every monolithic character of G is monomial, then G is solvable.*
- *If $|cd_m(G)| \leq 3$, then G is solvable.*

Strongly Monolithic Characters

Theorem (Erkoç, Güngör, Özkan 2022)

Let G be a solvable group and let p be a fixed prime number. Then the following hold:

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Theorem (Özkan, Erkoç 2022)

Let G be a group with at most three strongly monolithic characters. Then G is a solvable group.

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Let G be a nonabelian group. Then G is solvable if one of the following situations holds :

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- $|cd_{sm}(G)| \leq 2$.

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Let G be a nonabelian group. Then G is solvable if one of the following situations holds :

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- $\chi(1)$ *is a prime for every strongly monolithic character* χ *of G.*

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- *Every strongly monolithic character of G is monomial.*

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ρ_5	5	1	-1	0	0	orthogonal faithful

Literature

Definition (Qian, Wang,Wei, 2007)

Let G be a finite group and let χ be an irreducible character of G. The number $a(\chi) := |G : \ker\chi|/\chi(1)$ is said to be the codegree of the character χ.

Lemma (Qian, Wang,Wei, 2007)

Let G be a group and $\chi \in Irr(G)$. Then
(i)For any $N \trianglelefteq G$ with $N \leq \ker\chi$, χ may be viewed as a character of G/N , and the codegree $a(\chi)$ of χ is independent of the choice of such N.
(ii) If M is a subnormal subgroup of G and ψ is an irreducible constituent of χ_M, then $a(\psi)$ divides $a(\chi)$.

Literature

Theorem (Lu, Meng, 2024)

Let G be a solvable group. Then, $h(G) \leq |cod(G)| - 1$.

Strongly Monolithic Characters

Theorem (Erkoç, Yılmaztürk, 2024)

$h(G) \leq |cod_{msm}(G)| + 1$ for all finite solvable groups G.

$\mathrm{cod}_{\mathrm{msm}}(G) = \{\mathrm{a}(\chi) : \chi \in \mathrm{Irr}_{\mathrm{msm}}(G)\}$ where $\mathrm{Irr}_{\mathrm{msm}}(G)$ denote the set of monomial strongly monolithic characters.

Literature

Theorem (Lu, Meng 2024)

Let $\chi \in Irr(G)$ and $K = ker(\chi)$. Either of the following conditions guarantees the $\psi \in Irr(G)$ with $cod(\psi) > cod(\chi)$:
(1) $K \nsubseteq F(G)$;
(2) $K = F(G) < G$, and G/K is solvable, where $F(G)$ is the Fitting subgroup of G.

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Thank You...